<u>Continuity and Differentiability</u>

1. Let $f(x)=\begin{cases}\sin\frac{1}{x} & ,x\neq 0\\ 0 & ,x=0\end{cases}$. Is f(x) continuous ? Is f(x) differentiable ?

2. For n = 1, 2, …, let $f_n(x)=x^n\sin\frac{1}{x}$, $x\neq 0$ and $f_n(0)=0$.

Is $f_n(x)$ continuous ? Is $f_n(x)$ differentiable ?

3. Show that at x = 0

(i) $\frac{1-2^{1/x}}{1+2^{1/x}}$ has a jump discontinuity ;

(ii) $\frac{1}{3^{1/x}+1}$ has a jump discontinuity ;

(iii) $\frac{x}{3^{1/x}+1}$ has a removable discontinuity.

4. Let $f(x)=\begin{cases}e^{-\frac{1}{x^2}} & , & x\neq 0\\ 0 & , & x=0\end{cases}$. Prove that **(a)** $f'(0)=0$, **(b)** $f''(0)=0$.

5. Suppose that a function f is continuous on an interval **I** and that f(x) = 0 whenever x is rational, show that f(x) = 0 $\forall\, x\in$ **I** .

6. Let $f(x)=\begin{cases}x & , & x\text{ is rational}\\ 1-x & , & x\text{ is irrational}\end{cases}$ Dom (f) = [0, 1] .

(a) Show that f is continuous for only a x = ½ .

(b) Define g(x) = f(x) f(1 – x) . Where is g continuous ?

7. Prove that if f satisfies the equation f(x + y) = f(x) f(y) for all x and y , and if

f(x) = 1 + x g(x) where $\lim_{x\to 0} g(x)=1$, then $f'(x)$ exists for every x and $f'(x)=f(x)$.

8. If f satisfies the equation f(xy) = f(x) + f(y) for every real x , y ,

show that f(1) = 0 and f(1/x) = – f(x) .

If f is differentiable at each $x\neq 0$, show that $f'(x)=\frac{f'(1)}{x}$, for each $x\neq 0$.

9. Let $f_n(x) = x^n \sin\frac{1}{x}$. Find those n such that $f_n(x)$ is **(a)** continuous, **(b)** differentiable.

For those differentiable f_n's , calculate $\frac{d}{dx} f_n(x)$ and find those n such that $\frac{d}{dx} f_n(x)$ is continuous.

10. If $f(x)$ is differentiable, then, is it true that $f'(x)$ must be continuous ?

11. Let $C[a, b]$ denotes the set of all continuous functions from $[a, b]$ to $\mathbf{R}$, $C_1[a, b]$ be the set of all functions f from $[a, b]$ to $\mathbf{R}$ such that f' exists and is continuous.
Let $D_1[a, b]$ denotes the set of all differentiable functions from $[a, b]$ to $\mathbf{R}$, $D_2[a, b]$ be the set of all functions f from $[a, b]$ to $\mathbf{R}$ such that f'' exists.
Find the relations (in the sense of inclusion of sets) between $C[a, b]$, $C_1[a, b]$, $D_1[a, b]$ and $D_2[a, b]$.

12. Using the example $g(x) = \begin{cases} 1 & , x < 1 \\ 2 & , x > 1 \end{cases}$, $f(u) = \begin{cases} u & , u < 1 \\ 3u - 5 & , u > 1 \end{cases}$

to show that g is discontinuous at $x = x_0$, f discontinuous at $u_0 = g(x)$ does not imply $f(g(x))$ is discontinuous at x_0 .

13. Use the example $g(x) = \begin{cases} x & , x \le 1 \\ 5 & , x > 1 \end{cases}$, $f(u) = 9$

to show that g discontinuous at x_0 , f continuous at $u_0 = g(x_0)$ does not imply $f(g(x))$ is discontinuous at x_0 .

14. Using the example $g(x) = x^2$, $f(u) = \begin{cases} u & , u \le 1 \\ 3u - 5 & , u > 1 \end{cases}$

to show that g is continuous at $x = x_0$, f discontinuous at $u_0 = g(x)$ does not imply $f(g(x))$ is discontinuous at x_0 .